

# Are the neutrino masses and mixings closely related to the masses and mixings of quarks?<sup>1</sup>

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## Abstract

The mass matrices of quarks have a simple structure if expressed in powers of the small parameter  $\sigma = (m_c/m_t)^{1/2}$ . If there is a close relation between quarks and leptons, one would expect similar structures for the lepton matrices which involve the same parameter. To have a specific proposal, the see-saw mechanism is employed together with the stringent requirement that the singlet neutrinos carry non-zero generation charges. These charges then determine the powers of  $\sigma$  in the corresponding heavy neutrino mass matrix. As a result, the neutrino mixing matrix turns out to be of the bimaximal type. In addition, the mass splitting of the two lightest neutrinos is found to be tiny, and just of the correct magnitude necessary for the vacuum solution for solar neutrinos.

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# 1 Introduction

Recent experiments on solar and atmospheric neutrinos indicate flavor mixing of neutrinos, and finite neutrino masses [1], [2]. Nevertheless, our knowledge on the mass spectrum and the mixing of neutrinos is still very limited. Several different scenarios can be envisaged [3]. In contrast to the neutrino properties, the masses of the quarks and the pattern of their mixings are well known. On a qualitative level only the validity and shape of the unitarity triangle is still an open problem. In this situation one may ask whether or not there exists a close relation between the masses and mixings of leptons and those of quarks. If an intimate relation could be discovered it would greatly reduce the number of free parameters and would shed light on the new physics behind the standard model of particle physics. Such a relation cannot be a trivial one since the large mixing angle observed in the study of atmospheric neutrinos [2] has no counterpart in the quark sector.

The mass matrices of quarks, if expressed in powers of a small parameter, have a simple structure, suggestive of a new generation symmetry [4]. A connection between quarks and leptons would be manifest, if the mass matrices of leptons are governed by powers of the same small parameter. In this work a very close relation of this type will be proposed. To this end two general suggestions of grand unified theories [5] are used: the existence of 3 two-component neutrino fields which are singlets with respect to the standard model gauge groups, and the similarity of the Dirac neutrino and charged lepton mass matrices with the up-quark and the down-quark mass matrix, respectively [6], [7]. (But a full gauge theory ( $SO(10), E(6)$ ) with a necessarily complicated Higgs structure is not implied here, even though  $SO(10)$  models gave the first hint for a large neutrino mixing angle [6], [8]). The remaining and decisive mass matrix is then the one for the 3 singlet neutrinos. Since these fields are not protected by a standard model symmetry, their mass values will be very large.

Because of the Majorana type selfcoupling of the singlet neutrinos, their generation charges play an important role in restricting the structure of the corresponding mass matrix. After exploring these restrictions, the straightforward use of the see-saw mechanism finally gives a rather definite form for the mass matrix of the light neutrinos [9]. It has interesting properties: i) The mixing matrix obtained from it is of the bimaximal type and also includes CP-violating phases. ii) By fixing the scale for the singlet neutrinos to obtain the correct mass scale for the atmospheric neutrinos, the mass splitting of the two lightest neutrinos turns out to be tiny ( $\Delta m^2 \approx 10^{-11} - 10^{-10}(\text{eV})^2$ ), thus favoring the vacuum oscillation solutions [10] for solar neutrinos.

## 2 Quark mass matrices

The 3x3 mass matrices of up and down quarks contain 36 parameters, but only 10 parameters are physical: 6 eigenvalues, 3 mixing angles and 1 CP-violating phase.

Therefore, for different choices of independent parameters the mass matrices can look very different. From the point of view of assigning generation (family) charges to the quark fields it is preferable to use a basis in which each independent matrix element is represented by a different power of a small parameter. A natural choice for this parameter is  $\sigma = (m_c/m_t)^{1/2} = 0.058 \pm 0.004$ , because the top quark plays a dominant role among the quarks. Moreover, within experimental error limits, one finds for the up-quark mass ratios

$$m_u : m_c : m_t = \sigma^4 : \sigma^2 : 1 \quad (1)$$

and for the Kobayashi-Maskawa matrix elements [11]

$$|V_{us}| = 4\sigma, |V_{cb}| = \sigma/\sqrt{2} \quad , \quad |V_{ub}| = \sigma^2 \quad . \quad (2)$$

A suitable basis which involves just 5 parameters for the up-quark mass matrix  $m_U$  and 5 parameters for the down-quark mass matrix  $m_D$ , is to take  $m_U$  real and symmetric with  $(m_U)_{11} = 0$  and  $m_D$  hermitian with  $(m_D)_{13} = (m_D)_{31} = (m_D)_{23} = (m_D)_{32} = 0$ . The only complex matrix element is then  $(m_D)_{12} = (m_D)_{21}^*$ . The observed masses and mixing angles are well represented by taking

$$m_U = \begin{pmatrix} 0 & \sigma^3/\sqrt{2} & \sigma^2 \\ \sigma^3/\sqrt{2} & -\sigma^2/2 & \sigma/\sqrt{2} \\ \sigma^2 & \sigma/\sqrt{2} & 1 \end{pmatrix} m_t, \quad m_D = \begin{pmatrix} 0 & -i\sqrt{2}\sigma^2 & 0 \\ i\sqrt{2}\sigma^2 & -\sigma/3 & 0 \\ 0 & 0 & 1 \end{pmatrix} m_b \quad (3)$$

and  $\sigma = 0.057$ . However, for our purpose, it is convenient to transform to a basis in which  $m_U$  is diagonal and  $m_D = V m_D^{diagonal} V^\dagger$ , where  $V$  denotes the Cabibbo-Kobayashi-Maskawa matrix. To leading order in  $\sigma$  one finds

$$m_U = \begin{pmatrix} \sigma^4 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad m_D = \begin{pmatrix} O(\sigma^3) & -i\sqrt{2}\sigma^2 & \sigma^2 \\ i\sqrt{2}\sigma^2 & -\sigma/3 & \sigma/\sqrt{2} \\ \sigma^2 & \sigma/\sqrt{2} & 1 \end{pmatrix} m_b \quad . \quad (4)$$

For definiteness “maximal CP-violation” has been used [7, 12]. It is defined to maximize the area of the unitarity triangle with regard to phase changes of the off-diagonal elements<sup>3</sup>. It leads to a right-handed unitarity triangle with angles  $\alpha \approx 70^\circ$ ,  $\beta \approx 20^\circ$  and  $\gamma \approx 90^\circ$  [11, 12]. Irrespective of this choice the expressions (3) and (4) demonstrate that masses and mixings are governed by the same small parameter in a simple fashion.

### 3 The Dirac neutrino and charged lepton mass matrices

The 6x6 neutrino mass matrix (with zero entries for the light-light sector) has a block structure which contains a 3x3 Dirac neutrino mass matrix  $m_\nu^{Dirac}$  and the

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<sup>3</sup> $m_D$  in (4) can also be written in an equivalent form in which the off-diagonal matrix elements form an antisymmetric hermitian matrix [12].

3x3 mass matrix  $M$  for the 3 singlet fields  $\hat{\nu}_e, \hat{\nu}_\mu, \hat{\nu}_\tau$ <sup>4</sup>. Since there is no compelling theory which fixes these matrices and the charged lepton matrix  $m_E$ , one is forced to make assumptions. To have a very close connection between quark and leptons, and to be in line with the expectations from grand unified theories mentioned in section 1, I will assume that the mass matrices of quarks and leptons of the same isospin can be diagonalized simultaneously. For hermitian forms this implies

$$[m_\nu^{Dirac}, m_U] = 0 \quad [m_E, m_D] = 0 \quad . \quad (5)$$

More specifically, I will take

$$m_\nu^{Dirac} = m_U \quad (6)$$

and comment on deviations from (6) in section 5. These equations are assumed to hold at the scale of the heavy neutrinos.

Eq. (5) fixes the mass matrix for the charged leptons since  $m_D$  and the eigenvalues of  $m_E$  are known. To illustrate its form in terms of the parameter  $\sigma$ , one gets from  $m_e : -m_\mu : m_\tau = \frac{3}{2}\sigma^3 : -\sigma : 1$  and  $m_E = V m_E^{Diagonal} V^\dagger$  to leading order in  $\sigma$

$$m_E = \begin{pmatrix} O(\sigma^3) & -i3\sqrt{2}\sigma^2 & \sigma^2 \\ i3\sqrt{2}\sigma^2 & -\sigma & \sigma/\sqrt{2} \\ i\sigma^2 & \sigma/\sqrt{2} & 1 \end{pmatrix} m_\tau \quad . \quad (7)$$

Notably, in the basis in which the up-quark mass matrix is diagonal, the matrix for charged leptons contains complex elements like  $m_D$ .

## 4 The mass matrix for the singlet neutrinos

It remains to discuss the mass matrix  $M$  for the singlet neutrino fields  $\hat{\nu}_e, \hat{\nu}_\mu, \hat{\nu}_\tau$ . It determines the mass matrix for the light neutrinos  $m_\nu$  through the see-saw mechanism

$$m_\nu = -m_U \cdot M^{-1} \cdot m_U^T \quad . \quad (8)$$

This equation, in which (6) has already been used, is supposed to hold at the scale of the heavy neutrinos. Several suggestions for  $M$  can be found in the literature [13]. A close connection between charged and neutral fermions will exist if also  $M$ , in the basis in which  $m_U$  is diagonal, has a simple structure in terms of powers of the small parameter  $\sigma$  which occurs in (4), (6), and (7) [9]. These powers are restricted if the particle fields carry charges of a generation (family) symmetry [14].

Because of the self-coupling of the singlet fields, strong restrictions for the structure of  $M$  occur if the three  $U(1)$  generation charges are all different from zero (and  $M$  does not vanish in the limit  $\sigma \rightarrow 0$ ) [9]. By accepting this condition, two of the three fields must carry opposite  $U(1)$  charges. They then form, for  $\sigma \rightarrow 0$ , a

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<sup>4</sup>All fields are taken to be left-handed two-component fields.

heavy singlet neutrino of the Dirac type. Giving opposite charges to  $\hat{\nu}_e$  and  $\hat{\nu}_\tau$  and allowing integer powers of  $\sigma^2$  only, the matrix  $M$  has the structure<sup>5</sup>

$$M = \begin{pmatrix} \sim \sigma^6 & \sim \sigma^2 & 1 \\ \sim \sigma^2 & \sim \sigma^2 & \sim \sigma^4 \\ 1 & \sim \sigma^4 & \sim \sigma^6 \end{pmatrix} M_0. \quad (9)$$

The  $U(1)$  charges for  $\hat{\nu}_e, \hat{\nu}_\mu, \hat{\nu}_\tau$  are  $-3/2, 1/2, 3/2$ , respectively; they determine the powers of  $\sigma^2$  in (9). As in the case of the quark mass matrices, the proportionality factors multiplying the powers of  $\sigma$  should be of order 1. In particular, if the correlation with  $m_U$  is strict, the factor of  $\sigma^2$  in the first row and first column ( $p$  in eq. (10)) should be equal or very close to one. Because of the smallness of  $\sigma^4$  and  $\sigma^6$ ,  $M$  can be approximated by the simpler form

$$M = \begin{pmatrix} 0 & p\sigma^2 & 1 \\ p\sigma^2 & r\sigma^2 & 0 \\ 1 & 0 & 0 \end{pmatrix} M_0. \quad (10)$$

One can check that the approximation (10) is also applicable when calculating the mass matrix  $m_\nu$  for the light neutrinos according to (8), even though the inverse of the matrix  $M$  enters here. Moreover, a simple consideration of the original  $6 \times 6$  neutrino mass matrix (with zero entries in the light-light sector) shows that the coefficients  $p$  and  $r$  can be taken to be real.

## 5 The mass spectrum and the mixings of the light neutrinos

The consequences of the simple model considered here are now easily worked out. From Eqs. (4, 8, 10) one finds for the mass matrix of the light neutrinos

$$m_\nu = - \begin{pmatrix} 0 & 0 & r\sigma^2 \\ 0 & 1 & -p \\ r\sigma^2 & -p & p^2 \end{pmatrix} \frac{\sigma^2 m_t (M_0)^2}{r M_0}. \quad (11)$$

Apart from the mass scale  $M_0$  we are left with only two important parameters ( $p, r$ ). The remarkable point about this matrix is that it leads to a near degeneracy of the two lightest neutrinos, and for  $p \approx 1$ , to a neutrino mixing matrix which is of the bimaximal type as defined in [15]. To make contact with the atmospheric neutrino data [2] the heavy mass parameter  $M_0$  can be adjusted to give the heaviest of the light neutrinos ( $\nu_3$ ) the mass  $m_3 \approx 5.5 \times 10^{-2}$  eV. For  $p = r = 1$  one then finds

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<sup>5</sup>As long as we dismiss matrices  $M$  which lead to a vanishing determinant when neglecting higher powers of  $\sigma$  than  $\sigma^2$ , the form (9) is unique apart from its mirror form corresponding to the charges  $-3/2, -1/2, 3/2$ .

$M_0 = 10^{12}$  GeV,  $m_3^2 - m_1^2 = 3 \times 10^{-3}(\text{eV})^2$  and  $m_2^2 - m_1^2 = 1 \times 10^{-11}(\text{eV})^2$ . For  $r = 2$  one gets  $M_0 = 5 \times 10^{11}$  GeV and  $m_2^2 - m_1^2 = 7 \times 10^{-11}(\text{eV})^2$ . Thus, the neutrino mass matrix obtained by invoking an intimate relation between charged and neutral fermions favors large mixing angles for the atmospheric and for the solar neutrinos, and the vacuum oscillation solution [10] for the latter. But it seems not compatible with indications for  $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$  oscillations reported by the LSND collaboration [16].

For a more detailed treatment one has, of course, to take the charged lepton mass matrix  $m_E$ , which is nondiagonal in our basis, into account. In addition, the scale dependence of  $m_\nu$  according to the renormalization group has to be considered.

Transforming therefore to a basis in which  $m_E$  is diagonal, the corresponding neutrino mass matrix at the scale  $M_0$  is then

$$\tilde{m}_\nu(M_0) = V^T(M_0)m_\nu(M_0)V(M_0) . \quad (12)$$

For  $m_\nu(M_0)$  we have to replace (11) by

$$m_\nu(M_0) = - \begin{pmatrix} 0 & 0 & r \frac{m_u(M_0)}{m_c(M_0)} \\ 0 & 1 & -p \\ r \frac{m_u(M_0)}{m_c(M_0)} & -p & p^2 \end{pmatrix} \frac{m_c(M_0)m_t(M_0)}{rM_0} . \quad (13)$$

Now, the renormalization group equation for  $\tilde{m}_\nu(\mu)$  [17] can be used to calculate  $\tilde{m}_\nu$  at the weak scale  $m_Z$ . After this calculation the neutrino mixing matrix  $U$  is obtained by diagonalizing the hermitian matrix  $\tilde{m}_\nu \cdot \tilde{m}_\nu^*$ :

$$\tilde{m}_\nu(m_Z) \cdot \tilde{m}_\nu^*(m_Z) = U D D^* U^\dagger . \quad (14)$$

The diagonal matrix  $D$

$$D = U^\dagger \tilde{m}_\nu(m_Z) U^* . \quad (15)$$

gives the neutrino mass eigenvalues. These are complex because of the complex Cabibbo-Kobayashi-Maskawa matrix elements. Finally,  $U$  can be redefined by a diagonal phase matrix such that (15) contains now only positive definite eigenvalues. The new matrix  $U$  then expresses the light neutrino states  $\nu_e, \nu_\mu, \nu_\tau$  in terms of the neutrino mass eigenstates  $\nu_1, \nu_2, \nu_3$  according to

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = U \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} . \quad (16)$$

From the knowledge of  $U$ , the surviving and transition probabilities of the oscillating neutrino flavors can easily be calculated.

It turns out that the mass spectrum and mixings obtained from (14),(15) is not strikingly different from the ones found directly from the simple mass matrix, eq. (11). The small mixing angles occurring in  $V$  do not strongly influence  $\tilde{m}_\nu$ . The mixing matrix  $U$  has again – for  $p \approx 1$  – a near bimaximal form, but contains now

also CP-violating contributions [9]. Also the scale change from  $M_0 \approx 10^{12}$  GeV down to the weak scale does not lead to a qualitatively different picture. In particular, the near degeneracy of the two lightest neutrino masses remains stable [9] in contrast to cases discussed in ref. [18].

To illustrate the results I give here the actual numbers taking  $p = 1$  and  $r = 2$ . (They differ somewhat from the ones obtained in [9] since I use now in (12) the Cabibbo-Kobayashi-Maskawa Matrix  $V$  as a consequence of the condition (5)). With the input  $m_3 = 0.055$  eV one gets  $M_0 = 3 \times 10^{11}$  GeV,  $m_3^2 - m_1^2 = 3 \times 10^{-3}(\text{eV})^2$  and  $m_2^2 - m_1^2 = 9 \times 10^{-11}(\text{eV})^2$ . The survival and transition probabilities are (for notations see [9])

$$\begin{aligned}
P(\nu_e \rightarrow \nu_e) &= 1 - 0.95 S_{21} - 0.05 S_{31} - 0.04 S_{32} \\
P(\nu_\mu \rightarrow \nu_\mu) &= 1 - 0.32 S_{21} - 0.49 S_{31} - 0.49 S_{32} \\
P(\nu_\tau \rightarrow \nu_\tau) &= 1 - 0.21 S_{21} - 0.49 S_{31} - 0.50 S_{32} \\
P(\nu_e \rightarrow \nu_\mu) &= 0.53 S_{21} + 0.02 S_{31} + 0.02 S_{32} \\
&\quad + 0.07 T_{21} - 0.07 T_{31} + 0.07 T_{32} \\
P(\nu_e \rightarrow \nu_\tau) &= 0.42 S_{21} + 0.02 S_{31} + 0.03 S_{32} \\
&\quad - 0.07 T_{21} + 0.07 T_{31} - 0.07 T_{32} \\
P(\nu_\mu \rightarrow \nu_\tau) &= -0.21 S_{21} + 0.47 S_{31} + 0.47 S_{32} \\
&\quad + 0.07 T_{21} - 0.07 T_{31} + 0.07 T_{32} .
\end{aligned} \tag{17}$$

Our mass matrix for the light neutrinos is based on the eqs. (5), (6) and (13). If the eigenvalues of  $m_\nu^{Dirac}$  differ from those of the up-quark mass matrix, but (5) still holds, the corresponding modification can be absorbed into the parameters  $r$  and  $M_0$ , which have anyhow to be adjusted to the data. A large effective value for  $r$  would give a significantly larger value for  $m_2^2 - m_1^2$ . In this case the model predicts an energy independent deficit of solar neutrinos, which is, however, barely compatible with the results of the Homestake collaboration [1].

The bimaximal mixing depends on the parameter  $p$  and gets spoiled if  $p$  is sizeably different from 1. The mixing angle for atmospheric neutrinos is sensitive to  $p$ . Deviations from  $p = 1$  by more than 25 % would no more be compatible with the mixing angle extracted from present experiments.

## 6 Conclusion

A close relationship between leptons and quarks has been proposed. All mass matrices are expressed in terms of powers of the same small parameter  $\sigma = (m_c/m_t)^{1/2}$ . To obtain the neutrino mass matrix, the see-saw mechanism is employed. Rather detailed predictions on the spectrum and the mixing angles follow from an Ansatz for the mass matrix of the singlet neutrino fields, which is based on the assumption of non-zero generation (family) charges of these particles. The mixing matrix is of

the bimaximal type, and the mass difference of the two lightest neutrinos has the right magnitude required for the vacuum oscillation solution for solar neutrinos. The spectrum and the mixing pattern is stable with respect to loop corrections.

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